During my PhD I completed three papers, which have some overlapping themes but are on different topics within geometric group theory. These papers formed the main part of my thesis. My supervisor was Armando Martino, and I completed my PhD in September 2016.

Throughout my PhD I have enjoyed discussing problems with other PhD students and with members of staff. From my experience of being a teacher, I have aimed to present my research and related themes regularly. Because of this, I have given 20 academic talks during my 3 years as a PhD student. These have generally been hour or half hour long talks, and at either local seminars or at conferences that I have attended.

The first paper, on decision problems in groups. A decision problem generally asks for an algorithm which, for any valid input, correctly answers a fixed question with either a 'yes' or 'no'. We say that such a decision problem is Turing decidable if there is such an algorithm which can be performed on a Turing machine. Examples of decision problems for a group G with finite presentation $\langle S \mid R\rangle$ include:

(1) decide whether a given 'word' (an ordered sequence which takes values in $S \cup S^{-1}$) represents the identity element of G;
(2) decide whether two given words represent conjugate elements in G;
(3) decide whether a given presentation $\langle S' \mid R'\rangle$ represents a group isomorphic to G.

These problems are known respectively as the word problem, the conjugacy problem, and the isomorphism problem for the presentation $\langle S \mid R\rangle$, and were first discussed by Dehn [Deh11]. It is important to note that, if any of these problems is Turing decidable for one finite presentation of G, then it is Turing decidable for all finite presentations of G. It may come as a surprise that such simple looking problems cannot be solved for every finitely presented group, and moreover there exist explicit finite presentations with unsolvable word problem. This was the work of Novikov [Nov58], Boone [Boo59], and others.

What may also be surprising is that, if the conjugacy problem is solvable for a finitely presented group G, then it need not be solvable for finite index or finite extensions of G, even when these are of degree 2. My first task as a PhD student was to answer this question for the Houghton groups, which are a family of groups for which the conjugacy problem was solved relatively recently [ABM13]. This is an interesting family of groups since it does not fit into the main known classes, and so new techniques must be developed. The easiest Houghton group to describe is H_2. This naturally acts on the set $\mathbb{Z}$ with generating set consisting of the permutation $t : z \mapsto z + 1$ for all $z \in \mathbb{Z}$ and the transposition swapping 0 and 1. Each Houghton group has $\mathrm{FSym}(\mathbb{N})$, the group of all finitary permutations of the set $\mathbb{N}$, as a subgroup. This means that no Houghton group is residually finite. Also, each of these groups is *highly transitive* as defined by Osin [HO15]. Brown showed that, for each $n \geq 3$, H_n is F_{n-1} but not FP_n, where these are the well known homological finiteness conditions studied in relation to groups. It was in this first paper that I began to think of groups as collections of permutations, since this seems a profitable way of thinking about Houghton's groups. This work presents a natural question.

> *To what extent can the methods of* [ABM13] *be used for groups containing* FSym *as a subgroup? In particular, given an infinite group G with solvable conjugacy problem, let $\hat{G}$ denote the (right) regular representation of G. Then, does $\langle \hat{G}, \mathrm{FSym}(G)\rangle$ necessarily have solvable conjugacy problem? Setting G to be a free group of finite rank might be a sensible first case.*

The second paper, on the R_∞ property for groups. In my first paper I solved the twisted conjugacy problem for Houghton's groups. This is closely related to the R_∞ property which is satisfied by a group G if, for each $\phi \in \mathrm{Aut}(G)$, G has infinitely many ϕ-twisted conjugacy classes. I became aware of this problem after two papers simultaneously solved it for the Houghton groups and cited my first paper. I immediately saw a short proof that Houghton's groups have this property, which used the way that I had thought about these groups in my first paper. This was to think of the groups as permutations of a set X, meaning that these groups are subgroups of $\mathrm{Sym}(X)$, the group of all permutations of X. I therefore wondered whether the R_∞ property could be investigated for a much larger class of groups, those groups G for which there is an infinite set X such that $\mathrm{FSym}(X) \leqslant G \leqslant \mathrm{Sym}(X)$. Both the problem and the plans of attack were decided independently from my supervisor. My main result was that, for any infinite group G, the group $\langle \hat{G}, \mathrm{FSym}(G)\rangle$ has the R_∞ property. It is from my work on this paper that I know that I can work as an independent researcher. Future work relating to this paper could involve extending these methods. Currently many examples of groups $\mathrm{FSym}(X) \leqslant G \leqslant \mathrm{Sym}(X)$ are dealt with, and interesting corollaries are obtained, but the theorems do not apply to all such groups.

> *For an infinite set X, is it true that a group satisfying $\mathrm{FSym}(X) \leqslant G \leqslant \mathrm{Sym}(X)$ has the R_∞ property? In my viva Collin Bleak expressed interest in one particular such group which is bi-embeddable with Thompson's group V.*

The third paper, on the degree of commutativity of infinite groups. For a finite group F, the degree of commutativity is the probability of choosing two elements in F which commute i.e.

$$\frac{|\{(a, b) \in F^2 \ : \ ab = ba\}|}{|F|^2}.$$

In [AMV], this definition was generalised to any finitely generated infinite group G by using the Cayley graph of G with respect to a finite generating set S of G. This Cayley graph is locally finite, and so one can consider $\mathbb{B}_S(n)$, the ball of radius n with respect to S. A possible definition for the degree of commutativity of G with respect to S is therefore given by

$$\limsup_{n\to\infty} \frac{|\{(a, b) \in \mathbb{B}_S(n)^2 \ : \ ab = ba\}|}{|\mathbb{B}_S(n)|^2} \tag{1}$$

where the lim sup is used since it is unclear whether, in general, this will be a real limit. In [AMV] they conjecture that this limit will be positive if and only if the group is virtually abelian, and they verify this conjecture in the case of groups of subexponential growth and hyperbolic groups. I was asked to make further tests of the conjecture and so computed (1) for wreath products of the form $C \wr \mathbb{Z}$ where C is a cyclic group and for $F \wr \mathbb{Z}$ where F is a finite group.

Other future work. In my third paper I posed the following question. For a finite group F, the degree of commutativity is equal to

$$\frac{\text{the number of conjugacy classes in } F}{|F|}.$$

It therefore seemed natural to ask, for a finitely generated infinite group G, whether this could be taken as a definition for the degree of commutativity of G i.e. if

$$\limsup_{n\to\infty} \frac{\text{the number of conjugacy classes meeting } \mathbb{B}_S(n)}{|\mathbb{B}_S(n)|} \tag{2}$$

would satisfy the conjecture made in [AMV] and whether it is equivalent to the definition (1) above. My supervisor and I have organised a collaboration with Laura Ciobanu, who is currently at Heriot Watt University, in order to investigate these questions (partly since the numerator of (2) is the conjugacy growth function of G, a topic which Laura Ciobanu is currently working on).

References

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